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On Poreda's Problem on the Strong Unicity Constants

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1. INTRODUCTION

Let $C(I)$ denote the space of continuous, real-valued functions on the interval $I = [a, b]$ endowed with the uniform norm $\|\cdot\|$. Let Π_n denote the set of all algebraic polynomials of degree n or less, and let Π denote the set of all algebraic polynomials. For a given $f \in C(I)$, with best uniform approximation $T_n(f)$ from Π_n, Newman and Shapiro [10] showed that there is a constant $\gamma > 0$ such that

$$\|f - p\| \geqslant \|f - T_n(f)\| + \gamma \|p - T_n(f)\| \tag{1.1}$$

for all $p \in \Pi_n$. The largest such constant γ is written $\gamma_n(f)$ and is called the *strong unicity constant*. It is known that $0 < \gamma_n(f) \leqslant 1$ and that $\gamma_n(f) = 1$ for any function $f \in \Pi_n$. Let $M_n(f) = \gamma_n(f)^{-1}$. Properties of the sequence $\{M_n(f)\}_{n=0}^{\infty}$ have been studied in [7, 8, 11, 13]. In particular, in [11] Poreda asked:

> For what functions f in $C(I)$ is the sequence $\{M_n(f)\}_{n=0}^{\infty}$ bounded?

Let $B = \{f \in C(I): \{M_n(f)\}_{n=0}^{\infty}$ is bounded$\}$. Evidently, $\Pi \subseteq B$. Poreda [11] gave an example of a function $f \notin B$. Henry and Roulier [8] gave a wide class of functions which are not in B and conjectured that in fact $B = \Pi$.

Let $E_n(f) = \{x \in I: |f(x) - T_n(f)(x)| = \|f - T_n(f)\|\}$ be the set of extreme points of $f - T_n(f)$ and let $|E_n(f)|$ denote the cardinality of $E_n(f)$. Previous

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work indicates that the properties of the sequence $\{M_n(f)\}_{n=0}^{\infty}$ depend on the distribution of $E_n(f)$ in I and on $|E_n(f)|$. In particular, Schmidt [13] showed that if $|E_n(f)| = n+2$ for infinitely many n, then $f \notin B$. This raised the question of whether in fact there exists a nonpolynomial function f in $C(I)$ for which $|E_n(f)| > n+2$ for all but finitely many n. A negative answer to this question would solve Poreda's problem. In [13] it was also shown that if there is a nondegenerate interval $[c, d] \subseteq I$ for which $E_n(f) \cap [c, d] = \varnothing$ for infinitely many n, then $f \notin B$.

In Section 2 of this paper we demonstrate the existence of a nonpolynomial function f in $C(I)$ such that $|E_n(f)| > n+2$ for all n and show that B is of first category as is Π. In Section 3 we obtain an interpolatory lower estimate for $M_n(f)$ which is similar to an upper estimate for $M_n(f)$ given in [5]. This lower estimate will then be used in Section 4 to relax the condition $|E_n(f)| = n+2$ for infinitely many n in Theorem 4 of [13]. In Section 5, two conditions based on the distribution of $E_n(f)$ in I which ensure that $f \notin B$ are given.

2. Nonuniqueness of Alternants

For $f \in C(I)$, let $e_n(f) = f - T_n(f)$. Then $E_n(f) = \{x \in I\colon |e_n(f)(x)| = \|e_n(f)\|\}$.

Theorem 1. *There is a nonpolynomial $f \in C(I)$ such that $|E_n(f)| > n+2$ for all n.*

Proof. For convenience we assume that $I = [-1, 1]$. We show that there is an even function $f \in C(I)$ such that $0 \notin E_n(f)$ for $n = 0, 1, \ldots$. That such a function satisfies the conclusion of Theorem 1 can be seen as follows. Let α be the smallest positive element of $E_n(f)$. The number α exists since $0 \notin E_n(f)$ and $E_n(f)$ is compact. Since $e_n(f)$ is even, $e_n(f)(-\alpha) = e_n(f)(\alpha)$ and $E_n(f)$ contains no points in the open interval $(-\alpha, \alpha)$. Thus an alternant for $e_n(f)$ cannot include both α and $-\alpha$, and we see that $|E_n(f)| \geqslant n+3$.

We employ the Baire category theorem to demonstrate the existence of such a function. Let $\mathscr{E}$ be the closed subspace of $C(I)$ consisting of all even functions in $C(I)$. For $n = 0, 1, \ldots$, let

$$A_n = \{f \in \mathscr{E} : 0 \in E_n(f)\}.$$

To show that A_n is closed, let $\{f_k\}_{k=1}^{\infty}$ be a sequence in A_n and $f \in \mathscr{E}$ such that $\|f_k - f\| \to 0$ as $k \to \infty$. By the continuity of the operator T_n, $\|e_n(f_k)\| \to \|e_n(f)\|$ and $\|e_n(f_k)\| = |e_n(f_k)(0)| \to |e_n(f)(0)|$. Thus $|e_n(f)(0)| = \|e_n(f)\|$ and $0 \in E_n(f)$. So $f \in A_n$, and A_n is closed.

We now show that A_n has an empty interior. Let $f \in A_n$. We consider two cases.

Suppose $e_n(f)(0) = 0$. Then $f \in \Pi_n$. Given $\varepsilon > 0$ select $h \in \mathscr{C}$ so that $h(0) = 0$, $h(i/(n+2)) = (-1)^i \varepsilon$, $i = 1,..., n+2$, and h is linear on each of the intervals $[(i-1)/(n+2), i/(n+2)]$, $i = 1,..., n+2$. Now extend h to be even on $[-1, 1]$. Then $h \in \mathscr{C}$, $T_n(h) = 0$, and $E_n(h) = \{\pm i/(n+2): i = 1,..., n+2\}$. If $g = f + h$, then since $f \in \Pi_n$, $T_n(g) = f + T_n(h) = f$ and $e_n(g) = h$. As a result, $0 \notin E_n(g) = E_n(h)$, $g \notin A_n$, and $\|g - f\| = \|h\| = \varepsilon$. Hence, f is not an interior point of A_n.

Suppose $e_n(f)(0) \neq 0$. Without loss of generality, assume $\tau = e_n(f)(0) > 0$, and let ε, $0 < \varepsilon < \tau/2$, be given. Since $e_n(f)$ is continuous at 0, there is a $\delta > 0$ such that $0 \leqslant \tau - e_n(f)(x) < \varepsilon$ for $|x| < \delta$. Define h on $\{0, \delta/2, \delta\}$ by $h(0) = -\varepsilon$, $h(\delta/2) = \tau - e_n(f)(\delta/2)$, and $h(\delta) = 0$. Now extend h continuously to $[0, 1]$ so that $-\varepsilon \leqslant h(x) \leqslant \tau - e_n(f)(x)$ for $x \in [0, \delta]$, and $h(x) = 0$ for $x \in [\delta, 1]$. Finally, extend h to be even on $[-1, 1]$. Thus $h \in \mathscr{C}$ and $\|h\| = \varepsilon$. If we set $g = f + h$, then for $x \in [-1, -\delta] \cup [\delta, 1]$,

$$|g(x) - T_n(f)(x)| = |e_n(f)(x)| \leqslant \tau.$$

For $x \in [-\delta, \delta]$,

$$g(x) - T_n(f)(x) = e_n(f)(x) + h(x) \geqslant \tau - \varepsilon - \varepsilon > 0$$

and

$$g(x) - T_n(f)(x) = e_n(f)(x) + h(x) \leqslant e(f)(x) + \tau - e_n(f)(x) = \tau.$$

Moreover, $g(\delta/2) - T_n(f)(\delta/2) = \tau$. Thus $\|g - T_n(f)\| = \tau$. If we select an alternant for $e_n(f)$, then since $e_n(f) > 0$ on $[-\delta, \delta]$ at most one point in the alternant may lie in $[-\delta, \delta]$. Replacing this point by $\delta/2$, if necessary, we obtain an alternant for $g - T_n(f)$, and thus $T_n(g) = T_n(f)$. Since $g(0) - T_n(f)(0) = e_n(f)(0) - \varepsilon = \tau - \varepsilon$, $0 \notin E_n(g)$ and $g \notin A_n$. In addition, $\|g - f\| = \|h\| = \varepsilon$. Hence, f is not in the interior of A_n, and so A_n has an empty interior. By the Baire category theorem, $\mathscr{C} \neq \bigcup_{n=0}^{\infty} A_n$, and the proof of Theorem 1 is complete.

The proof of Theorem 1 shows the existence of a set of functions of second category in $\mathscr{C}$ for which Poreda's question remains unanswered. In contrast to this, we have:

THEOREM 2. *B is of first category in $C(I)$.*

Proof. For $L = 1, 2,...,$ let $B_L = \{f \in C(I): M_n(f) \leqslant L,\ n = 0, 1,...\}$. By Theorem 2 in [1], $M_n(f)$ is a lower semicontinuous function of f for each n, and B_L is closed. Let $f \in B_L$. For each n, select a polynomial g_n of exact degree $n+1$ such that $\|f-g_n\|\to 0$ as $n\to\infty$. By Theorem 2 in [7], $\lim_{n\to\infty} M_n(g_n)=\infty$, and thus every neighborhood of f contains a function not in B_I. B_I has an empty interior, and Theorem 2 is proven.

3. Interpolatory Estimates for $M_n(f)$

For fixed n, let $S_n = \{p\in \Pi_n : \|p\|=1\}$. The following characterization of $\gamma_n(f)$ is due to Bartelt and McLaughlin (see [1] and Theorem 5 in [3]). If $f\in C(I)\backslash\Pi_n$, then

$$\gamma_n(f)=\min_{p\in S_n}\max_{x\in E_n(f)} |\operatorname{sgn} e_n(f)(x)|\, p(x). \tag{3.1}$$

Many of the analyses of the asymptotic behavior of $M_n(f)$ rely on an interpolatory characterization of $M_n(f)$ (see [7, 8, 13]). Let

$$x_0 < x_1 < \cdots < x_{n+1}$$

be an alternant for $e_n(f)$. For $j=0,\dots,n+1$, let q_j be the unique polynomial in Π_n satisfying $q_j(x_i)=\operatorname{sgn} e_n(f)(x_i)$, $i=0,\dots,n+1$, $i\neq j$. Cline [4] has shown that

$$K_n = K_n(x_0,\dots,x_{n+1}) = \max_{0\leqslant j\leqslant n+1} \|q_j\| \tag{3.2}$$

is a suitable strong unicity constant, that is, $K_n \geqslant M_n(f)$. Henry and Roulier [8] proved that $K_n = M_n(f)$ if $|E_n(f)| = n+2$. As a result, the analyses of [7, 8, 13] either impose the condition or conditions which imply $|E_n(f)|=n+2$. Unfortunately, when $|E_n(f)|>n+2$, we only have

$$M_n(f)\leqslant \min K_n, \tag{3.3}$$

where the minimum in (3.3) is taken over all alternants for $e_n(f)$. An example which appears in [2] shows that the inequality in (3.3) can be strict when $|E_n(f)|>n+2$.

In this section, we obtain a lower estimate for $M_n(f)$ when $|E_n(f)|\geqslant n+2$ and $T_n(f)\neq T_{n+1}(f)$.

Lemma 1. *If $f\in C(I)\backslash\Pi_n$, then*

$$M_n(f)=\max\{\|p\| : p\in\Pi_n,\ \sigma(x)\,p(x)\leqslant 1 \text{ for } x\in E_n(f)\}, \tag{3.4}$$

where $\sigma(x)=\operatorname{sgn} e_n(f)(x)$.

Proof. The assertion on p. 64 of Rice [12] (note the misprint: = should be $\geqslant$) implies that the maximum in (3.4) exists. Let $q\in\Pi_n$ satisfy

$\sigma(x)\,q(x) \leqslant 1$ for $x \in E_n(f)$ and $\|q\| = \max\{\|p\|: p \in \Pi_n,\ \sigma(x)\,p(x) \leqslant 1$ for $x \in E_n(f)\}$. Then $q/\|q\|$ has norm 1 and by (3.1)

$$M_n(f)^{-1} = \gamma_n(f) \leqslant \max_{x \in E_n(f)} \sigma(x)\,q(x)/\|q\| \leqslant 1/\|q\|$$

and thus $\|q\| \leqslant M_n(f)$. By (3.1) again, select $p \in S_n$ such that $M_n(f)^{-1} = \gamma_n(f) = \max_{x \in E_n(f)} \sigma(x)\,p(x)$. Then $\sigma(x)\,p(x)\,M_n(f) \leqslant 1$ for $x \in E_n(f)$, and $M_n(f) = \|pM_n(f)\| \leqslant \|q\|$. Hence, $\|q\| = M_n(f)$, and Lemma 1 is proven.

Suppose that $T_n(f) \neq T_{n+1}(f)$. Then $e_n(f)$ can demonstrate no more than $n+2$ points of alternation in $E_n(f)$. We may decompose $E_n(f)$ into $n+2$ nonempty subsets

$$E^0, E^1, \ldots, E^{n+1} \tag{3.5}$$

satisfying (i) E^i is compact, $i = 0, \ldots, n+1$, (ii) $\max E^i < \min E^{i+1}$, $i = 0, \ldots, n$, (iii) $\sigma(x) = \operatorname{sgn} e_n(f)(x)$ is constant over E^i, $i = 0, \ldots, n+1$, and (iv) $\operatorname{sgn} e_n(f)(x)|_{E^i} = -\operatorname{sgn} e_n(f)(x)|_{E^{i+1}}$, $i = 0, \ldots, n$.

If $|E_n(f)| = n+2$, then each E^i is a singleton, and the q_j in (3.2) are well defined. If $|E_n(f)| \geqslant n+2$, we demonstrate the existence of analogous interpolating polynomials.

LEMMA 2. *Suppose that $T_n(f) \neq T_{n+1}(f)$, E^i, $i = 0, \ldots, n+1$, are given by* (3.5)*, and $\sigma(x)$ is defined in* (iii) *above. Then for $j = 0, \ldots, n+1$, there is a $q_{nj} \in \Pi_n$ and there are points $y^i_{nj} \in E^i$, $i = 1, \ldots, n+1$, $i \neq j$, such that $\sigma(x)\,q_{nj}(x) \leqslant 1$ for $x \in E_n(f)$ and $\sigma(y^i_{nj})\,q_{nj}(y^i_{nj}) = 1$, $i = 0, \ldots, n+1$, $i \neq j$. Moreover, each q_{nj} is unique in the sense that q_{nj} is the only polynomial in Π_n such that $\sigma(x)\,q_{nj}(x) \leqslant 1$ for $x \in E_n(f)$ and $\sigma(x)\,q_{nj}(x) = 1$ for some $x \in E^i$, $i = 0, \ldots, n+1$, $i \neq j$.*

Proof. In this proof we suppress the subscripts on q_{nj} and y^i_{nj}. This result depends not on f but on $E_n(f)$ and its decomposition (3.5).

We first consider the case in which $E_n(f)$ is finite and induct on $|E_n(f)|$. If $|E_n(f)| = n+2$, then we can write $E^i = \{x_i\}$, $i = 0, \ldots, n+1$. For fixed j, let $y^i = x_i$, $i = 0, \ldots, n+1$, $i \neq j$, and let q be the polynomial in Π_n satisfying $q(x_i) = \sigma(x_i)$, $i = 0, \ldots, n+1$, $i \neq j$. Then $\sigma(x_j)\,q(x_j) \leqslant 0$, for otherwise q would have $n+1$ zeros. Thus the conclusion of Lemma 2 holds if $|E_n(f)| = n+2$.

Assume that the conclusion of Lemma 2 holds whenever $|E_n(f)| = m \geqslant n+2$. Let $|E_n(f)| = m+1$, and fix j, $0 \leqslant j \leqslant n+1$. If $|E^j| \geqslant 2$, delete one point z from E^j and apply the induction hypothesis to obtain $q \in \Pi_n$ and $y^i \in E^i$, $i = 0, \ldots, n+1$, $i \neq j$, such that $\sigma(x)\,q(x) \leqslant 1$ for $x \in E_n(f) \setminus \{z\}$ and $\sigma(y^i)\,q(y^i) = 1$, $i = 0, \ldots, n+1$, $i \neq j$. As before, $\sigma(z)\,q(z) \leqslant 0$, and the result holds. Suppose $|E^j| = 1$. Then for some k, $0 \leqslant k \leqslant n+1$, $k \neq j$, $|E^k| \geqslant 2$. Delete one point z from E^k and apply the induction hypothesis to obtain

$q \in \Pi_n$, $y^i \in E^i$, $i = 0, \ldots, n+1$, $i \neq k$, $i \neq j$, and $y^k \in E^k \backslash \{z\}$ such that $\sigma(x)\, q(x) \leqslant 1$ for $x \in E_n(f) \backslash \{z\}$ and $\sigma(y^i)\, q(y^i) = 1$, $i = 0, \ldots, n+1$, $i \neq j$. If $\sigma(z)\, q(z) \leqslant 1$, then the result holds. Now suppose that $\sigma(z)\, q(z) > 1$. Delete y^k from E^k and apply the induction hypothesis again to obtain $\bar{q} \in \Pi_n$, $\bar{y}^i \in E^i$, $i = 0, \ldots, n+1$, $i \neq k$, $i \neq j$, and $\bar{y}^k \in E^k \backslash \{y^k\}$ such that $\sigma(x)\, \bar{q}(x) \leqslant 1$ for $x \in E_n(f) \backslash \{y^k\}$ and $\sigma(\bar{y}^i)\, \bar{q}(\bar{y}^i) = 1$, $i = 0, \ldots, n+1$, $i \neq j$. We show that $\sigma(y^k)\, \bar{q}(y^k) \leqslant 1$. Suppose $\sigma(y^k)\, \bar{q}(y^k) > 1$. Then $\sigma(z)[q(z) - \bar{q}(z)] > 0$ and $\sigma(y^k)[q(y^k) - \bar{q}(y^k)] < 0$. Since $\sigma(z) = \sigma(y^k)$, $q - \bar{q}$ has a zero between z and y^k. Similarly, $q - \bar{q}$ has a zero between y^i and $\bar{y}^i$ (possibly inclusive), $i = 0, \ldots, n+1$, $i \neq k$, $i \neq j$. Thus $q - \bar{q}$ has $n+1$ zeros and must vanish identically. So $q = \bar{q}$ and $\sigma(y^k)\, \bar{q}(y^k) = \sigma(y^k)\, q(y^k) \leqslant 1$. Thus the conclusion of Lemma 2 holds for $|E_n(f)|$ finite.

Now suppose $|E_n(f)|$ is infinite.

Since each E^i is compact, we may select $n+2$ sequences $\{E^i_k\}_{k=1}^{\infty}$, $i = 0, \ldots, n+1$, of nonempty, finite sets such that $E^i_k \subseteq E^i$ and

$$d(E^i_k, E^i) = \sup_{y \in E^i} \inf_{x \in E^i_k} |x - y| \to 0$$

as $k \to \infty$, $i = 0, \ldots, n+1$. Fix j, $0 \leqslant j \leqslant n+1$. For each k, we obtain $q_k \in \Pi_n$ and $y^i_k \in E^i_k$, $i = 0, \ldots, n+1$, $i \neq j$, such that $\sigma(x)\, q_k(x) \leqslant 1$ for $x \in E_k = \bigcup_{i=0}^{n+1} E^i_k$ and $\sigma(y^i_k)\, q_k(y^i_k) = 1$, $i = 0, \ldots, n+1$, $i \neq j$. Since each E^i is compact, we may pass to a subsequence and relabel so that $y^i_k \to y^i \in E^i$ as $k \to \infty$, $i = 0, \ldots, n+1$, $i \neq j$. By Lemma 3 in [9], the q_k are uniformly bounded, and thus another relabeling allows us to assume that $q_k \to q \in \Pi_n$ uniformly on I as $k \to \infty$. For $x \in E_n(f)$, say $x \in E^i$, we may find a sequence $\{x_k\}_{k=1}^{\infty}$, where each $x_k \in E^i_k$ and $x_k \to x$ as $k \to \infty$. Thus $\sigma(x)\, q(x) = \lim_{k \to \infty} \sigma(x_k)\ q(x_k) \leqslant 1$. Also, $\sigma(y^i)\ q(y^i) = \lim_{k \to \infty} \sigma(y^i_k) \cdot q_k(y^i_k) = 1$, $i = 0, \ldots, n+1$, $i \neq j$. Thus the existence of the q_{nj} and the y^i_{nj} is demonstrated. The proof of the uniqueness of the q_{nj} is the same as part of the induction step in the first case (that in showing $\bar{q} = q$), and we omit this detail.

THEOREM 3. *Suppose that $f \in C(I) \backslash \Pi_n$ and that $T_n(f) \neq T_{n+1}(f)$. Let E^i, $i = 0, \ldots, n+1$, be the decomposition of $E_n(f)$ given in* (3.5), *and let q_{nj}, $j = 0, \ldots, n+1$, be the unique polynomials given in Lemma* 2. *Then*

$$M_n(f) \geqslant \max_{0 \leqslant j \leqslant n+1} \|q_{nj}\|.$$

Proof. The lower estimate (3.6) follows directly from the inequality $\sigma(x)\, q_{nj}(x) \leqslant 1$ for $x \in E_n(f)$ and Lemma 1.

An example can easily be constructed for which the inequality in (3.6) is strict.

4. CONDITIONS ON $|E_n(f)|$

For $f \in C(I)\backslash \Pi$, let $\{n_k\}_{k=1}^{\infty}$ be the strictly increasing sequence of nonnegative integers whose range contains precisely those n for which $T_n(f) \neq T_{n+1}(f)$. For each k, let

$$E^0, E^1, \ldots, E^{n_k+1} \tag{4.1}$$

be the decomposition of $E_{n_k}(f)$ given by (3.5) with $n = n_k$.

THEOREM 4. *Let $f \in C(I)\backslash \Pi$ and let $\{n_k\}_{k=1}^{\infty}$ be as described above. If for infinitely many k, at most two of the sets E^i, $i = 0, \ldots, n_k + 1$, contain more than one point, then $f \notin B$.*

Proof. By relabeling we may assume that at most two of the E^i contain more than one point for all $k = 1, 2, \ldots$. For each j, $0 \leqslant j \leqslant n_k + 1$, let the polynomials q_{kj} and the points y_{kj}^i, $i = 0, \ldots, n_k + 1$, $i \neq j$, be as in Lemma 2. For convenience, let $y_{kj}^{-1} = a$ and $y_{kj}^{n+2} = b$. It is possible that $y_{kj}^{-1} = y_{kj}^0$ or $y_{kj}^{n+2} = y_{kj}^{n+1}$.

We first assert that, after extracting a subsequence and relabeling, for each k there exists j_k, $0 \leqslant j_k \leqslant n_k + 1$, and a polynomial $p_k \in \Pi_{n_k}$ such that $|p_k(y_{kj_k}^i)| \leqslant 1$, $i = 0, \ldots, n_k + 1$, $i \neq j_k$, and

$$\lim_{k \to \infty} \max_{x \in J_k} |p_k(x)| = \infty, \tag{4.2}$$

where $J_k = [y_{kj_k}^{j_k - 1}, y_{kj_k}^{j_k + 1}]$.

For each k, let μ_k and ν_k be such that $|E^{\mu_k}| \geqslant 1$ and $|E^{\nu_k}| \geqslant 1$ and $|E^i| = 1$, $i = 0, \ldots, n_k + 1$, $i \neq \mu_k$, $i \neq \nu_k$. For fixed k, we can find a polynomial $P_k \in \Pi_{n_k}$ such that $|P_k(y_{k\mu_k}^i)| = 1$, $i = 0, \ldots, n_k + 1$, $i \neq \mu_k$, and

$$\|P_k\| \geqslant \frac{2}{\pi} \log(n_k - 1) - c, \tag{4.3}$$

where c is an absolute constant. The polynomial P_k is obtained by removing absolute value signs and inserting appropriate factors in the terms of the Lebesgue function corresponding to the nodes $y_{k\mu_k}^i$, $i = 0, \ldots, n_k + 1$, $i \neq \mu_k$. Inequality (4.3) then follows from the results of Erdös [6]. Thus $\lim_{k \to \infty} \|P_k\| = \infty$.

If $\max\{|P_k(x)|: x \in E^{\mu_k}\}$ is unbounded, we relabel further so that $\lim_{k \to \infty} \max\{|P_k(x)|: x \in E^{\mu_k}\} = \infty$. In this case, we let $j_k = \mu_k$ and $p_k = P_k$.

Suppose that $\max\{|P_k(x)|: x \in E^{\mu_k}\} \leqslant A$ for all k where $A \geqslant 1$. If $\max\{|P_k(x)|: x \in E^{\nu_k}\}$ is unbounded, then a relabeling allows us to assume that $\lim_{k \to \infty} \max\{|P_k(x)|: x \in E^{\nu_k}\} = \infty$. Since $y_{k\nu_k}^i = y_{k\mu_k}^i$ for $i \neq \nu_k$ and $i \neq \mu_k$, we see that $|P_k(y_{k\nu_k}^i)| \leqslant A$ for $i = 0, \ldots, n_k + 1$, $i \neq \nu_k$. In this case, we let $j_k = \nu_k$ and $p_k = P_k/A$.

Finally suppose that $|P_k(x)| \leqslant B$ for all $x \in E_{n_k}(f)$ and $k = 1, 2,...,$ where $B \geqslant 1$. For each k, select $x_k \in I$ such that $|P_k(x_k)| = \|P_k\|$. Since $y_{kj}^{j+1} \in E^{j+1}$ and $y_{k(j+1)}^{j} \in E^{j}$, $y_{k(j+1)}^{j} < y_{kj}^{j+1}$ for $j = 0,..., n_k + 1$ and so

$$I = \bigcup_{j=0}^{n+1} [y_{kj}^{j+1}, y_{kj}^{j+1}].$$

In this case, we select j_k so that $x_k \in [y_{kj_k}^{j_k+1}, y_{kj_k}^{j_k+1}]$ and let $p_k = P_k/B$. The first assertion is now established.

Assume now that, after relabeling, j_k and $p_k \in \Pi_{n_k}$ have been chosen for each k so that (4.2) and that above (4.2) hold. For $i = 0,..., n_k + 1, i \neq j_k$, let l_{ki} be the polynomial in Π_{n_k} such that $l_{ki}(y_{kj_k}^i) = 1$ and $l_{ki}(y_{kj_k}^j) = 0, j = 0,..., n_k + 1, j \neq i, j \neq j_k$. For $i = 0,..., n_k + 1, i \neq j_k$, the polynomials $\sigma(y_{kj_k}^i)\, l_{ki}(x)$ have the same sign on the interval $\operatorname{int}(J_k)$. Now select $x_k \in J_k$ such that $|p_k(x_k)| = \max_{x \in J_k} |p_k(x)|$. Then using the Lagrange interpolation formula, we have

$$\begin{aligned} |p_k(x_k)| &= \left| \sum_{\substack{i=0 \\ i \neq j_k}}^{n_k+1} p_k(y_{kj_k}^i)\, l_{ki}(x_k) \right| \leqslant \sum_{\substack{i=0 \\ i \neq j_k}}^{n_k+1} |l_{ki}(x_k)| \\ &= \left| \sum_{\substack{i=0 \\ i \neq j_k}}^{n_k+1} \sigma(y_{kj_k}^i)\, l^{ki}(x_k) \right| = |q_{kj_k}(x_k)| \leqslant M_{n_k}(f) \end{aligned}$$

by Theorem 3. It now follows that $\lim_{k \to \infty} M_{n_k}(f) = \infty$ and $f \notin B$.

The following corollary is a special case of Theorem 4 with more concrete conditions and follows at once.

COROLLARY 1. *Let $f \in C(I) \backslash \Pi$. If $|E_n(f)| \leqslant n + 4$ for infinitely many n, then $f \notin B$.*

Corollary 1 extends Theorem 4 in [13] and now reduces the resolution of Poreda's problem to considering those $f \in C(I)$ such that $|E_n(f)| > n + 4$ for all but finitely many n.

In [13] it was observed that if $f^{(n+1)} \neq 0$ in the open interval (a, b) for infinitely many n, then $f \notin B$ because such a function satisfies $|E_n(f)| = n + 2$.

COROLLARY 2. *Let $f \in C(I) \cap C^{\infty}(a, b)$. If $f^{(n)}$ vanishes at most five times in (a, b) for infinitely many n, then $f \notin B$.*

Proof. Let n be such that $f^{(n+1)}$ vanishes at most five times in (a, b). Let m be the largest nonnegative integer such that $T_m(f) = T_n(f)$. Then $T_m(f) \neq$

$T_{m+1}(f)$ and let $E^0, \ldots, E^{m+1}$ be the decomposition of $E_m(f)$ given by (3.5). We show that at most two of the sets E^i, $i = 0, \ldots, m+1$, contain more than one point. If $E^i = \{x\}$, then $e_m(f)'(x) = 0$ unless x is a or b. If $\{x, y\} \subseteq E^i$, then $e_m(f)'(x) = e_m(f)'(y) = 0$ and $e_m(f)'(z) = 0$ for some z between x and y unless x or y is a or b. If more than two of the sets E^i contain more than one point, then $e_m(f)'$ must have at least $m+6$ zeros in (a, b). By Rolle's Theorem $f^{(n+1)} = e_m(f)^{(n+1)}$ must have at least $m+6-n$ zeros in (a, b). Since $m \geqslant n$, $f^{(n+1)}$ would have at least six zeros in (a, b) which is a contradiction.

5. Conditions on the Distribution of $E_n(f)$

For convenience in this section, let $I = [0, \pi]$, $P_n = \text{span}\ \{1, \cos x, \ldots, \cos nx\}$, and $P = \bigcup_{n=0}^{\infty} P_n$. Let $f \in C[0, \pi] \backslash P$. Since $f \notin P$, there is a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of nonnegative integers whose range consists precisely of those n for which $T_n(f) \neq T_{n+1}(f)$. Here $T_n(f)$ denotes the best uniform approximation to f from P_n. As in (3.5) we decompose $E_{n_k}(f) = \{x \in [0, \pi] : |e_{n_k}(f)(x)| = \|e_{n_k}(f)\|\}$, where $e_{n_k}(f) = f - T_{n_k}(f)$ into the subsets $E^0, E^1, \ldots, E^{n_k+1}$. In order to describe the distribution of $E_{n_k}(f)$ in $[0, \pi]$, let $\alpha_i = \min E^i$, $\beta_i = \max E^i$, $i = 0, \ldots, n_k + 1$, and let

$$\Delta_{n_k} = \max\{\beta_i - \alpha_i : i = 0, \ldots, n_k + 1\}$$

and

$$\delta_{n_k} = \max\{\alpha_1 ; \pi - \beta_{n_k} ; \alpha_{i+1} - \beta_{i-1}, i = 1, \ldots, n_k\}.$$

Then Δ_{n_k} is the largest diameter of the sets E^i, and δ_{n_k} is the diameter of the largest interval in $[0, \pi]$ which contains one E^i but no other points in $E_n(f)$. Observe that $\Delta_{n_k} < \delta_{n_k}$. We establish conditions on Δ_{n_k} and δ_{n_k} which ensure that $f \notin B$.

Theorem 5. *If* $\liminf_{k \to \infty} n_k \Delta_{n_k} = 0$, *then* $f \notin B$.

Proof. Let $S_k = \{p \in P_{n_k} : \|p\| = 1\}$. Then by (3.1) applied to trigonometric polynomials,

$$\gamma_{n_k}(f) = \min_{p \in S_k} \max_{x \in E_{n_k}(f)} |\text{sgn}\, e_{n_k}(f)(x)|\, p(x). \tag{5.1}$$

By a remark of Henry and Roulier [8]

$$K_{n_k}{}^1 = \min_{p \in S_k} \max_{0 \leqslant i \leqslant n_k + 1} |\text{sgn}\, e_{n_k}(f)(\alpha_i)|\, p(\alpha_i). \tag{5.2}$$

where $K_{n_k} = K_{n_k}(\alpha_0, \ldots, \alpha_{n_k+1})$ is given by (3.2) using trigonometric polynomials. Select $q \in S_k$ such that

$$K_{n_k}^{-1} = \max_{0 \leqslant i \leqslant n_k+1} |\operatorname{sgn} e_{n_k}(f)(\alpha_i)| \, q(\alpha_i)$$

and select $y \in E_{n_k}(f)$ such that

$$|\operatorname{sgn} e_{n_k}(f)(y)| \, q(y) = \max_{x \in E_{n_k}(f)} |\operatorname{sgn} e_{n_k}(f)(x)| \, q(x).$$

Now select j such that $y \in E^j$. Then $\operatorname{sgn} e_{n_k}(f)(y) = \operatorname{sgn} e_{n_k}(f)(\alpha_j)$ and

$$\begin{aligned} &|\operatorname{sgn} e_{n_k}(f)(y)| \, q(y) - |\operatorname{sgn} e_{n_k}(f)(\alpha_j)| \, q(\alpha_j) \\ &\quad = |q(y) - q(\alpha_j)| = |q'(\xi)| \, |y - \alpha_j| \leqslant n_k \Delta_{n_k} \end{aligned} \tag{5.3}$$

for some ξ between α_j and y, where the last inequality follows from Bernstein's inequality [4, p. 91]. Thus by (5.1), (5.2), and (5.3)

$$\begin{aligned} \gamma_{n_k}(f) &\leqslant \max_{x \in E_{n_k}(f)} |\operatorname{sgn} e_{n_k}(f)(x)| \, q(x) = |\operatorname{sgn} e_{n_k}(f)(y)| \, q(y) \\ &\leqslant |\operatorname{sgn} e_{n_k}(f)(\alpha_j)| \, q(\alpha_j) + n_k \Delta_{n_k} \\ &= K_{n_k}^{-1} + n_k \Delta_{n_k}. \end{aligned}$$

By Theorem 2 in [13], $K_{n_k}^{-1} \to 0$ as $k \to \infty$, and by hypothesis $n_r \Delta_{n_r} \to 0$ as $r \to \infty$ for a subsequence $\{n_r\}$ of $\{n_k\}$. Thus $\gamma_{n_r}(f) \to 0$ as $r \to \infty$, $\{M_n(f)\}_{n=0}^{\infty}$ is unbounded, and $f \notin B$.

THEOREM 6. *If* $\limsup_{k \to \infty} n_k \delta_{n_k} = \infty$, *then* $f \notin B$.

Proof. Assume without loss of generality that $\lim_{k \to \infty} n_k \delta_{n_k} = \infty$. The proof is given for $\delta_{n_k} = \alpha_{j+1} - \beta_{j-1}$ since the case $\delta_{n_k} = \alpha_1$ and $\delta_{n_k} = \pi - \beta_{n_k}$ are similar. Let $c = (\beta_{j-1} + \alpha_{j+1})/2$. Define $h \in C[0, \pi]$ by $h(0) = h(\beta_{j-1}) = h(\alpha_{j+1}) = h(\pi) = 0$, $h(c) = -\operatorname{sgn} e_{n_k}(f)|_{E^j}$, and linear in between these points. Then h satisfies a Lipschitz condition with constant $\lambda = 2\delta_{n_k}^{-1}$. By Jackson's Theorem [4, p. 143] there is a polynomial $p \in P_{n_k}$ such that $\|h - p\| \leqslant \pi\lambda/2(n_k+1) = \pi/(n_k+1)\,\delta_{n_k}$. If $\tau_k = \pi/(n_k+1)\,\delta_{n_k}$, then $\lim_{k \to \infty} \tau_k = 0$. For k sufficiently large, $\|p\| \geqslant \|h\| - \tau_k = 1 - \tau_k > 0$. For $x \in E_{n_k}(f) \backslash E^j$,

$$|\operatorname{sgn} e_{n_k}(f)(x)| \, p(x) \leqslant |p(x)| = |h(x) - p(x)| \leqslant \tau_k.$$

Since $\operatorname{sgn} h(x) = -\operatorname{sgn} e_n(f)(x)$, for $x \in E^j \subseteq (\beta_{j-1}, \alpha_{j+1})$ we have

$$\begin{aligned} |\operatorname{sgn} e_n(f)(x)| \, p(x) &\leqslant |\operatorname{sgn} e_n(f)(x)| (p(x) - h(x)) \\ &\leqslant |p(x) - h(x)| \leqslant \tau_k. \end{aligned}$$

By (5.1)

$$\gamma_{n_k}(f) \leqslant \max_{x \in E_{n_k}(f)} |\operatorname{sgn} e_n(f)(x)|\, p(x)/\|p\|$$

$$\leqslant \tau_k/\|p\| \leqslant \tau_k/(1-\tau_k).$$

Thus $M_{n_k}(f) \geqslant (1-\tau_k)/\tau_k$ which tends to ∞ as $k \to \infty$, and $f \notin B$.

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